Lecture 3: The Diamond Model

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Outline

- Critique of the Solow-Swan model
- Environments of the Diamond model
  - Individuals
  - Firms
  - Markets
  - Timing of events
- Competitive equilibruim
  - Firms' profit maximisation
  - Individuals' utility maximisation

Critique of the Solow-Swan Model (1 of 2)

- Merits
  - The model identifies several important sources of growth: capital accumulation (through saving and investment), and technological progress.
  - It leads to a number of testable prediction that stand up quite well in empirical testing.
  - It proposed a basic framework for neoclassical growth model.
  - It is one of the most widely applied models in economics, instructing a lot of empirical studies on the determinants of growth (e.g. growth accounting, growth regressions).

Critique of the Solow-Swan Model (2 of 2)

- Criticism
  - The model takes the saving rates as exogenous and fixed.
  - Treatment of technology is highly incomplete.
    - The model does not identify what the "effectiveness of labour" is.
    - Technological progress, the only force giving rise to per-capita growth in long term, is exogenous.
  - The model does not incorporate human capital, which is also viewed as important for growth.

What is Next?

- Extending the Solow model
- Endogenizing the saving rate: The Diamond model and the Ramsey model
  - Both model continue to take the growth rates of L and A as given, but investigate the determinants of saving and investment from decision at the microeconomic level.
  - The two model present two basic framework for modeling the macroeconomy.
    - The Diamond model (Diamond(1965)): OLG framework
    - The Ramsey model: Representative agent framework
- New growth theory
  - Identify A and endogenize its evolution
  - Incorporate human capital and model its accumulation.

Motivation of OLG Model

- Neoclassical growth model relies on the representative household.
- Diamond model allows for the arrival of new households into the economy: realistic (takes account of life cycle), but also introduces a range of new economic interactions. Decisions by older generations affect the prices faced by younger generations.

The Baseline OLG model

- The discussion of the Diamond model gives you an illustration of the neoclassical approach.
- **Time** is discrete and runs to infinity, $t = 0, 1, 2, ...$
- **Individuals**:
  - The economy is populated with 2-period lived individuals:
    - Those born at t lives at t and $t + 1$ only.
  - In period t, L_t new individuals are born and L_{t-1} old individuals are dying (two generations overlap). Assume population grows at rate n:

$$L_t = (1 + n)L_{t-1} \tag{1}$$

  - The initial old refers to the generation who are already in their old age when time starts ($t = 0$). All other generations are called future generations, who live for 2 periods.

Utility (1 of 2)

- **Preference**: The lifetime utility of an individual born at t is given by:

$$U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1}) \tag{2}$$

  - where c_{1t} and c_{2t+1} denote the consumption when young and when old (utility of an initial old is simply $u(c_2, 0)$)
  - U is additively separable.
  - $\beta\,(0 < \beta < 1)$ is the discount factor.
  - General assumptions on the periodic utility function u:
    - u is increasing $(u'(c) > 0)$
    - u is concave $(u''(c) < 0)$
    - u satisfies the Inada condition to ensure positive consumption in equilibrium $(\lim_{c\to 0} u'(c) = +\infty)$

Utility (2 of 2)

- To make some tight predictions, we need to put more structure on utility and production function (next).
- A widely used form u takes is the constant relative risk aversion (CRRA) utility function

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta} = ln(c) \text{ if } \theta = 1 \tag{3}$$

  - Proof see Appendix 1
  - where $\theta > 0$ is the coefficient of relative risk aversion ($\frac{-cu''(\theta)}{u'(\theta)} = \theta$), the higher θ is, the more risk averse individuals are. It can be shown that $\frac{1}{\theta}$ is the elasticity of substitution between consumption in the two periods (see Appendix 2).

Endowment

- Each young individual is endowed with 1 unit of labour, which is supplied inelastically to firms.
- There is some initial capital stock K_0 owned equally by all initial old individuals.

Firms: Production Function

- In each period, firms hire labour and rent capital from individuals to produce output, sell the output in good market, and pay labour and capital. The CRS production function

$$Y_t = F(K_t, A_t L_t^D) \tag{4}$$

  - The intensive form of production: $y_t = f(k_t)$, where $k_t \equiv \frac{K_t}{A_t L_t^D}$. Assume that $f'(k) > 0, f''(k) < 0$ and $\lim_{k \to 0} f'(k) = +\infty$.
  - A_t grows at exogenous rate g.
  - There is no depreciation of capital.
  - With CRS, the number of firms can be normalized to 1.

Markets and Initial Conditions

- Markets are perfectly competitive.
- Denote the real interest rate (capital rental rate) and the real wage rate per unit of effective labour in period t as r_t and w_t, respectively.
- **Initial conditions**: A_0, L_{t-1} (number of initial old), K_0 are given.

In any given period t

- **Timing** of events in period t
  - Young individuals are born with labour endowments.
  - Firms hire labour from young individuals and rent capital from old individuals to produce output. Capital and labour are paid.
  - Old individuals consume capital income and existing wealth, then die and exit the economy.
  - Young individuals divide labour income between first-period consumption and saving to period $t+1$ which form new capital for period $t+1$ production.

The Competitive Equilibrium (Solution of the Model)

- **Firm's profit maximisation**
- Problem: The representative firm chooses capital demand and labour demand to maximize profits, taking interest rate and wages as given

$$\max_{K_t, L_t^D} F(K_t, A_t L_t^D) - r_t K_t - w_t A_t L_t^D$$

Recall that $F(K_t, A_t L_t^D) = A_t L_t^D f(k_t)$, $K_t = A_t L_t^D k_t$, thus

$$\max_{K_t, L_t^D} A_t L_t^D [f(k_t) - r_t k_t - w_t]$$

FOCs (for choice variables)

$$w.r.t\ k_t:\ r_t = f'(k_t) \tag{5}$$

$$w.r.t\ L_t^D:\ w_t = f(k_t) - f'(k_t) k_t \tag{6}$$

Firm's Problem (Cont'd)

- In fact, $f'(k_t)$ is the marginal product of capital ($MPK \equiv \frac{\partial F(K_t, A_t L_t^D)}{\partial K_t}$), and $f(k_t) - f'(k_t)k_t$ is the marginal product of effective labour ($\frac{\partial F(K_t, A_t L_t^D)}{\partial (A_t L_t^D)}$) (see Appendix 3 for proof)
- Firms' profit maximisation implies that factors of production earn their marginal products – a property of competitive markets.
- Notice that firms earn zero economic profits – a property of competitive markets.
- CRS can greatly simplifies firms' behavior. No need to worry about firm entry/exit, ownership of firms and dividends policy.
- Empirically, aggregate production function is close to CRS.

Individual's Problem (1 of 6)

- **Individual's utility maximisation**
- Situations faced by individuals:
  - The initial old simply rent the endowed capital to firms and consume the rental income and non-depreciated capital.
  - All future generations face the same situations: In the first period of life, supply labour and divide labour income between first-period consumption and saving (by holding capital). In the second period, simply consume the saving and any interests earned.

Individual's Problem (2 of 6)

- The only decision individuals need to make is for young individual to decide how much to consume today and how much to save for old ages, to maximize lifetime utility subject to budget constraints, taking prices (rational expectation applies) and A_t as given:

$$\max_{s_t} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$$

s.t.

$$c_{1t} + s_t = w_t A_t \quad (7)$$

$$c_{2t+1} = (1 + r_{t+1}) s_t \quad (8)$$

  - where Eqs.(7) and (8) are the individual's budget constraints when young and when old. Note that the wage rate per worker or per unit of labour is $w_t A_t$.
  - FOC: $\frac{\partial U(c_{1t}, c_{2t+1})}{\partial s_t} = 0$

Individual's Problem (3 of 6)

$$\frac{\partial U(c_{1t}, c_{2t+1})}{\partial s_t} = U_1(c_{1t}, c_{2t+1})\frac{\partial c_{1t}}{\partial s_t} + U_2(c_{1t}, c_{2t+1})\frac{\partial c_{2t+1}}{\partial s_t} = 0$$
$$= u'(c_{1t})(-1) + \beta u'(c_{2t+1})(1 + r_{t+1}) = 0$$

Therefore,

$$\frac{u'(c_{1t})}{\beta u'(c_{2t+1})} = 1 + r_{t+1} \tag{9}$$

- Eq.(9) is called the consumption Euler equation, a necessary condition for individual's intertemporal consumption choice. Notice that

$$\frac{u'(c_{1t})}{\beta u'(c_{2t+1})} = \frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = MRS_{c_{1t}, c_{2t+1}}$$

- Hence, Eq.(9) states that to maximise the consumer lifetime utility, the marginal rate of substitution between current and further consumption must equal to the gross interest rate.

Individual's Problem (4 of 6)

- The second order condition (SOC) is given by:

$$\frac{\partial^2 U}{\partial s_t^2} = u''(c_{1t}) + \beta u''(c_{2t+1})(1 + r_{t+1})^2 < 0$$

  - Hence, there is a unique s_t that solves the individual's problem.
  - In summary, combining the Euler equation (9) and the budget constraints (7) and (8) gives us the equation that determines the unique solution s_t:

$$\frac{u'(w_t A_t - s_t)}{\beta u'[(1 + r_{t+1})s_t]} = 1 + r_{t+1} \tag{10}$$

  - Eq.(10) determines the saving s_t as a function of r_{t+1}, w_t, A_t and parameters, which the individuals takes as given. c_{1t} amd c_{2t+1} are then determined by Eqs.(7) and (8), i.e. the individual's problem is solved!

Individual's Problem (5 of 6)

- Another way to solve the individual utility maximisation problem is to solve a constrained maximisation problem.

$$\max_{c_{1t}, c_{2t+1}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$$
$$s.t. \quad c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = A_t w_t \tag{11}$$

  - where Eq.(11) is an individual's lifetime budget constraint. It is obtained from the periodic budget constraints (7) and (8).
  - Notice the similarity between this formulation and a standard utility maximisation problem in microeconomics.

$$\max_{x_1, x_2} U(x_1, x_2) \quad s.t. \quad p_1 x_1 + p_2 x_2 = w$$

  - Here the two goods are first-period consumption and second-period consumption. The price for second-period consumption in terms of first-period consumption is $\frac{1}{(1+r_{t+1})}$.

Individual's Problem (6 of 6)

- Recall that FOC for the standard utility maximisation problem is given by:

$$\frac{U_1(x_1, x_2)}{U_2(x_1, x_2)} = \frac{p_1}{p_2}$$

- The above equation states that marginal rate of substitution between the two goods is equal to their price ratio. The FOC for our problem is given by (see Appendix 4 for proof):

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = 1 + r_{t+1}$$

- Notice that this equation is consistent with the standard form above: $(x_1 = c_{1t},\ x_2 = c_{2t+1},\ p_1 = 1,\ p_2 = \frac{1}{1+r_{t+1}})$

Appendix 1

Prove that $\lim_{\theta\to 1}\frac{c^{1-\theta}-1}{1-\theta} = ln(c)$

Proof.

Note that $\lim_{\theta\to 1}(c^{1-\theta}-1) = 1-1 = 0$, $\lim_{\theta\to 1}(1-\theta) = 0$. Therefore the limit is of "$\frac{0}{0}$" type. We need to use the L'Hôspital's rule to solve it. That is, we differentiate the numerator and the denominator with respect to θ separately, then compute the limit of the ratio of the two derivatives. We first do a bit of transformation for the limit:

$$\lim_{\theta\to 1}\frac{c^{1-\theta}-1}{1-\theta} = \lim_{\theta\to 1}\frac{e^{ln(c^{1-\theta})}-1}{1-\theta} = \lim_{\theta\to 1}\frac{e^{(1-\theta)ln(c)}-1}{1-\theta}$$

Apply L'Hôspital's rule:

$$\lim_{\theta\to 1}\frac{e^{(1-\theta)ln(c)}-1}{1-\theta} = \lim_{\theta\to 1}\frac{e^{(1-\theta)ln(c)}[-ln(c)]}{-1} = ln(c)$$

□

Appendix 2

Show $\frac{1}{\theta}$ in the CRRA utility function is the elasticity of substitution between consumption in the two periods.

Solution: Since the first derivative of the CRRA utility function is

$$u'(c) = c^{-\theta}$$

then, the marginal rate of substitution (MRS) is:

$$\frac{u'(c_1)}{u'(c_2)} = \frac{c_1^{-\theta}}{c_2^{-\theta}} = (\frac{c_2}{c_1})^{\theta}$$

solving for $\frac{c_2}{c_1}$, get:

$$\frac{c_2}{c_1} = (\frac{u'(c_1)}{u'(c_2)})^{\frac{1}{\theta}}$$

Hence, $\frac{1}{\theta}$ is the elasticity of the ratio of consumption in the two periods with respect to MRS. By definition, $\frac{1}{\theta}$ is then the elasticity of substitution, which is constant for the CRRA utility function.

Appendix 3

Prove that $\frac{\partial F(K_t, A_t L_t^D)}{\partial K_t} = f'(k_t)$ and $\frac{\partial F(K_t, A_t L_t^D)}{\partial A_t L_t^D} = f(k_t) - f'(k_t)k_t$

Proof.

Recall that $F(K_t, A_t L_t^D) = A_t L_t^D f(k_t)$, and $k_t = \frac{K_t}{A_t L_t^D}$, therefore:

$$\frac{\partial F(K_t, A_t L_t^D)}{\partial K_t} = A_t L_t^D f'(k_t) \frac{\partial k_t}{\partial K_t} = A_t L_t^D f'(k_t) \frac{1}{A_t L_t^D} = f'(k_t)$$

$$\begin{aligned}
\frac{\partial F(K_t, A_t L_t^D)}{\partial A_t L_t^D} &= f(k_t) + A_t L_t^D f'(k_t) \frac{\partial k_t}{\partial A_t L_t^D} \\
&= f(k_t) + A_t L_t^D f'(k_t) [-\frac{K_t}{(A_t L_t^D)^2}] \\
&= f(k_t) - f'(k_t) \frac{K_t}{A_t L_t^D} \\
&= f(k_t) - f'(k_t) k_t
\end{aligned}$$

□

Appendix 4 (1 of 2)

Find the Euler equation for the following problem:

$$\max_{c_{1t}, c_{2t+1}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$$

$$s.t. \quad c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = A_t w_t$$

Solution: Set up the Lagrangian function:

$$\mathcal{L} = U(c_{1t}, c_{2t+1}) + \lambda(A_t w_t - c_{1t} - \frac{1}{1 + r_{t+1}} c_{2t+1})$$

where λ is the Lagrangian multiplier. FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = U_1(c_{1t}, c_{2t+1}) - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = U_2(c_{1t}, c_{2t+1}) - \lambda \frac{1}{1 + r_{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = A_t w_t - c_{1t} - \frac{1}{1 + r_{t+1}} c_{2t+1} = 0$$

Appendix 4 (2 of 2)

Then, we have:

$$U_1(c_{1t}, c_{2t+1}) = \lambda$$

$$U_2(c_{1t}, c_{2t+1}) = \lambda \frac{1}{1 + r_{t+1}}$$

It is apparently that:

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = 1 + r_{t+1}$$

This is the consumption Euler equation. Together with the lifetime budget constraint, you get 2 equations with two unknowns (c_{1t} and c_{2t+1}). Then, the individual's problem is solved.